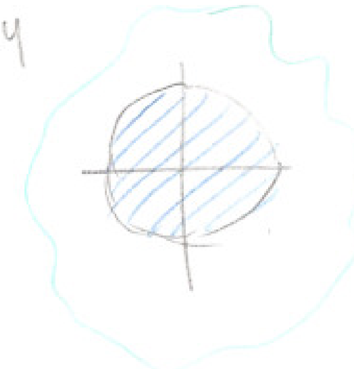
LAST TIME: SSR of Discrete time Systems

ie - Stability



note:

- can not be described by D.E.
- Use difference equations

SSR (Discrete Time)

$$x^{(k+1)} = F x^{(k)} + G u^{(k)}$$

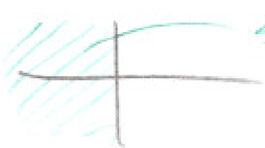
$$u = -K x^{(k)}$$

$$x^{(k+1)} = \underbrace{(F - GK)}_{\text{eig values must be } <1} x^{(k)}$$

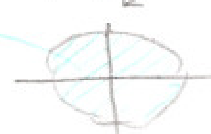
SSR (Time domain)

$$\left.x((k+1)T) = F x(kT) + G u(kT)\right\} t = (k+1)T$$

Cont. Time	Descrete time
$\dot{x} = Ax + Bu$	$x_{k+1} = F x_k + G u_k$



stability region



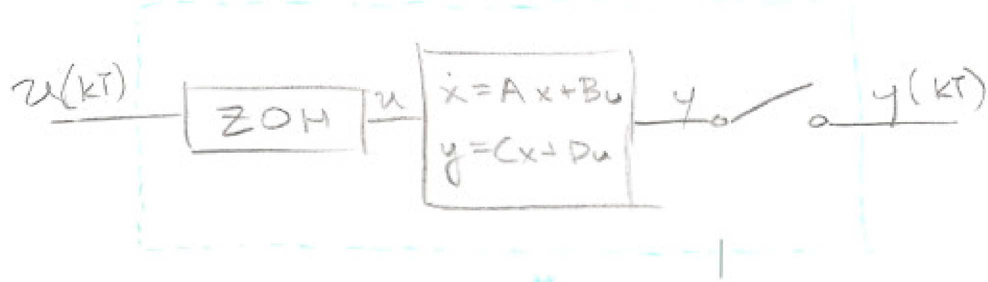
Laplace Transform

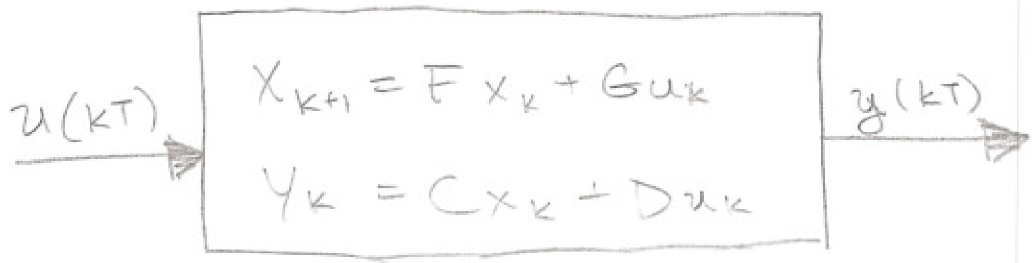
$$\frac{Y(s)}{U(s)} = G(s)$$

$$\boxed{G(s) \neq G(z)}$$

Z transform

$$\frac{Y(z)}{U(z)} = G(z)$$





$$\boxed{\begin{aligned} y(z) &= \mathcal{Z}(y(kt)) \\ u(z) &= \mathcal{Z}(u(kt)) \end{aligned}}$$

Z - Transform

- Given a sequence $f_0, f_1, \dots f_k$

The z-transform of the sequence f_k

$$\boxed{F(z) = \mathcal{Z}(f_k) = \sum_{k=0}^{\infty} f_k z^{-k}}$$

- z is a complex variable (equivalent to laplace varible s)

Ex: $\quad f(t) = e^{-at} \Rightarrow \mathcal{L} \Rightarrow \frac{1}{s+a} = F(s)$

Now if: $f_0 = 1, \; f_1 = e^{-at}, \; f_2 = e^{-2at}, \dots$

$$f_k = e^{-kat}, \quad k \in I$$

$$z(f_k) = F(z) = \sum_{k=0}^{\infty} e^{-akt} z^{-k}$$

$$= \sum_{k=0}^{\infty} \left[\left(z e^{at} \right)^{-1} \right]^k$$

$$F(z) = \frac{1}{1-(ze^{at})^{-1}} = \frac{1}{1 - z^{-1}e^{-at}}$$

$$= \frac{ze^{at}}{ze^{at} - 1}$$

geometric series summation.

Properties of Z-transform:

* $z(f(k)) = F(z)$

* $z(\alpha_1 f_1(k) + \alpha_2 f_2(k)) = \alpha_1 F_1(z) + \alpha_2 F_2(z)$

* $z(f_{(k+1)}) = zF(z) - zf(0)$

very similiar to time shift in L.T.

* $z(f_{(k+l)}) = z^l F(z) - z \sum_{i=1}^{l} z^{l-i} f(i-1)$ Adv

* $z(f_{(k-1)}) = z^{-1} F(z) + f(-1)$

* $z(f_{(k-l)}) = z^{-l} F(z) + \sum_{i=1}^{\infty} z^{-l+i} f(-i)$ delay.

* $z(a^k f(k)) = F(z/a)$

* $z(n f(k)) = -z \dfrac{df(z)}{dz}$

$$* \quad \mathcal{Z}\left(\sum_{\ell=0}^{\infty} f(\ell)\, g(k-\ell)\right) = F(z)\, G(z)$$

Final Value theorem

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (1 - z^{-1}) F(z)$$

given $F(z)$ has no $|\text{poles}| \geq 1$

TRANSFER FUNCTIONS

$$x_{k+1} = F x_k + G u_k$$
$$y_k = C x_k + D u_k$$

$$\mathcal{Z}(x_{k+1}) = F\, \mathcal{Z}(x_k) + G\, \mathcal{Z}(x_k)$$

given no initial conditions

$$z\, x(z) = F\, x(z) + G\, u(z)$$

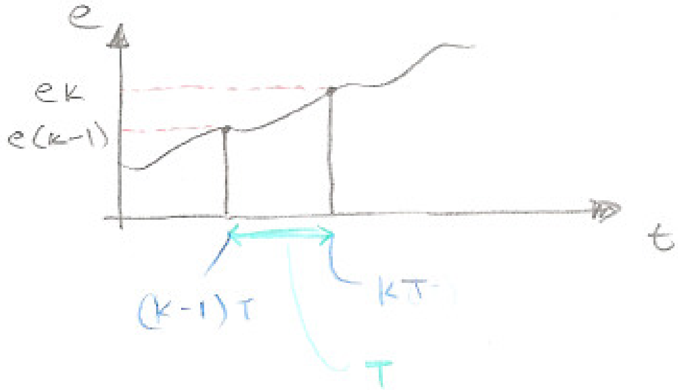
$$(zI - F)\, x(z) = G\, u(z)$$

$$x(z) = (zI - F)^{-1} G\, u(z)$$

$$Y(z) = C\, (zI - F)^{-1} G\, u(z) + D\, u(z)$$

$$\boxed{\frac{Y(z)}{u(z)} = C\, (zI - F)^{-1} G + D}$$

Origin of the Difference Equation



$$u_k = u_{k-1} + e_{k-1}T + \tfrac{1}{2}(e_k - e_{k-1})T$$

$$u_k = u_{k-1} + \frac{T}{2}\left[e_k + e_{k-1}\right]$$

z transform

$$U(z) = z^{-1}U(z) + \frac{T}{2}\left[E(z) + z^{-1}E(z)\right]$$

$$(1-z^{-1})U(z) = \frac{T}{2}(1+z^{-1})E(z)$$

$$\frac{U(z)}{E(z)} = \frac{T}{2}\,\frac{(1+z^{-1})}{(1-z^{-1})}$$

approx compaired to $1/s$.

FROM TF. TO S.S.R

$$\frac{U(z)}{E(z)} = \frac{b_0 z^3 + b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}$$

$$U(z) = \frac{b(z)}{a(z)}E(z) = b(z)\,Q(z)$$

$$\frac{E(z)}{a(z)}$$

$$(z^3 + a_1 z^2 + a_2 z + a_3)\, Q(z) = E(z) \qquad ①$$

$$(b_0 z^3 + b_1 z^2 + b_2 z + b_3)\, Q(z) = U(z) \qquad ②$$

① $\Rightarrow$ $$z^3 Q(z) + a_1 z^2 Q(z) + a_2 z Q(z) + a_3 Q(z) = E(z)$$

$$q(k+3) + a_1 q(k+2) + a_2 q(k+1) + a_3 q(k) = e(k)$$

$$q(k) = z^{-1}\{Q(z)\} \qquad e_k = z^{-1}\{E(z)\}$$

let:

$$x_1(k) = q(k+2) \qquad x_3(k+1) = x_2$$

$$x_2(k) = q(k+1) \qquad x_2(k+1) = x_1$$

$$x_3(k) = q(k) \qquad x_1(k+1) = q(k+3)$$

$$x_1(k+1) = e(k) - a_1 x_1(k) - a_2 x_2(k) - a_3 x_3(k)$$